DAY FOURTY

Mock Test 3

(Based on Complete Syllabus)

Instructions

1. This question paper contains of 30 Questions of Mathematic, divided into two Sections : **Section A** Objective Type Questions and **Section B** Numerical Type Questions.
2. Section A contains 20 Objective questions and all Questions are compulsory (**Marking Scheme** : Correct +4, Incorrect −1).
3. Section B contains 10 Numerical value questions out of which only 5 questions are to be attempted (**Marking Scheme** : Correct +4, Incorrect 0).

Section A : Objective Type Questions

1 Two circles in complex plane are

$C_1 : |z - i| = 2$

$C_2 : |z - 1 - 2i| = 4$. Then,

(a) C_1 and C_2 touch each other
(b) C_1 and C_2 intersect at two distinct points
(c) C_1 lies within C_2
(d) C_2 lies within C_1

2 If $\int \frac{\sqrt{\cos 2x}}{\sin x} dx = -\log|\cot x + \sqrt{\cot^2 x - 1}| + A + C$,

then A is equal to

(a) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right|$

(b) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \sec^2 x}}{\sqrt{2} - \sqrt{1 - \sec^2 x}} \right|$

(c) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \sin^2 x}}{\sqrt{2} - \sqrt{1 - \sin^2 x}} \right|$

(d) None of the above

3 If m_1 and m_2 are the roots of the equation $x^2 + (\sqrt{3} + 2)x + \sqrt{3} - 1 = 0$, then the area of the triangle formed by the lines $y = m_1 x$, $y = -m_2 x$ and $y = 1$ is

(a) $\frac{1}{2}\left(\frac{\sqrt{3} + 2}{\sqrt{3} - 1}\right)$
(b) $\frac{1}{2}\left(\frac{\sqrt{3} + 2}{\sqrt{3} + 1}\right)$
(c) $\frac{1}{2}\left(\frac{-\sqrt{3} + 2}{\sqrt{3} - 1}\right)$
(d) None of these

4 The equation of the ellipse whose axes are coincident with the coordinate axes and which touches the straight lines $3x - 2y - 20 = 0$ and $x + 6y - 20 = 0$, is

(a) $\frac{x^2}{5} + \frac{y^2}{8} = 1$
(b) $\frac{x^2}{40} + \frac{y^2}{10} = 10$
(c) $\frac{x^2}{40} + \frac{y^2}{10} = 1$
(d) $\frac{x^2}{10} + \frac{y^2}{40} = 1$

5 A variable straight line through the point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ meets the coordinate axes in A and B.

I. The locus of mid-point of AB is the curve $2xy(a + b) = ab(x + y)$.
II. The locus of mid-point of AB is the curve $2xy(x + y) = ab(a + b)$.

(a) Both I and II are true
(b) Only I is true
(c) Only II is true
(d) Both I and II are false

6 If the function $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$, $x\neq 0$ is continuous at each point of its domain, then the value of $f(0)$ is

(a) 2 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{1}{3}$

7 If two distinct chords, drawn from the point (p,q) on the circle $x^2+y^2=px+qy$, (where $pq\neq 0$) are bisected by the X-axis, then

(a) $p^2=q^2$ (b) $p^2=8q^2$
(c) $p^2<8q^2$ (d) $p^2>8q^2$

8 If A and B are different matrices satisfying $A^3=B^3$ and $A^2B=B^2A$, then

(a) det (A^2+B^2) must be zero
(b) det $(A-B)$ must be zero
(c) det (A^2+B^2) as well as det $(A-B)$ must be zero
(d) Atleast one of det (A^2+B^2) or det $(A-B)$ must be zero

9 A fair coin is tossed 100 times. The probability of getting tails 1, 3, ..., 49 times is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

10 The distance between the line

$$r=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+3\hat{\mathbf{k}}+\lambda(\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})$$

and the plane $r\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$ is

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{10}{3\sqrt{3}}$ (d) $\frac{10}{9}$

11 The negation of the compound preposition $p\vee(\sim p\vee q)$ is

(a) $(p\wedge\sim q)\wedge\sim p$ (b) $(p\wedge\sim q)\vee\sim p$
(c) $(p\wedge\sim q)\vee\sim p$ (d) $(p\wedge q)\wedge q$

12 If $f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5}, & \text{for } x\neq 1\\ -1/2, & \text{for } x=1\end{cases}$, then $f'(1)$ is equal to

(a) $-\frac{1}{9}$ (b) $-\frac{2}{9}$ (c) $-\frac{1}{3}$ (d) $\frac{1}{3}$

13 The value of $\tan\left\{\cos^{-1}\left(\frac{-2}{7}\right)-\frac{\pi}{2}\right\}$ is

(a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

14 A boat is being rowed away from a cliff of 150 m height. At the top of the cliff the angle of depression of boat changes from 60° to 45° in 2 min. Then, the speed of the boat (in m/h) is

(a) $\frac{4500}{\sqrt{3}}$ (b) $\frac{4500}{\sqrt{3}}(\sqrt{3}-1)$
(c) $\frac{4300}{\sqrt{3}}$ (d) $\frac{4500}{\sqrt{3}}(\sqrt{3}+1)$

15 The equation of a line of intersection of planes $4x+4y-5z=12$ and $8x+12y-13z=32$ can be written as

(a) $\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z}{4}$ (b) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$
(c) $\frac{x}{2}=\frac{y+1}{3}=\frac{z-2}{4}$ (d) $\frac{x}{2}=\frac{y}{3}=\frac{z-2}{4}$

16 If the value of the determinant $\begin{vmatrix}a&1&1\\1&b&1\\1&1&c\end{vmatrix}$ is positive, then

(a) $abc>1$ (b) $abc>-8$
(c) $abc<-8$ (d) $abc>-2$

17 A bag contains a white and b black balls. Two players A and B alternately draw a ball from the bag, replacing the ball each time after the draw. A begins the game.

If the probability of A winning (that is drawing a white ball) is twice the probability of B winning (that is drawing a black ball), then the ratio $a:b$ is equal to

(a) 1 : 2 (b) 2 : 1
(c) 1 : 1 (d) None of these

18 If $(5+2\sqrt{6})^n=I+f;\ n,I\in N$ and $0\le f<1$, then I equals

(a) $\frac{1}{f}-f$ (b) $\frac{1}{1+f}-f$ (c) $\frac{1}{1+f}+f$ (d) $\frac{1}{1-f}-f$

19 The median of a set of 11 distinct observations is 20.5. If each of the last 5 observations of the set is increased by 4, then the median of the new set

(a) is increased by 2
(b) is decreased by 2
(c) is two times the original a median
(d) remains the same as that of the original set

20 Solution of the differential equation $\dfrac{\sqrt{x}\,dx+\sqrt{y}\,dy}{\sqrt{x}\,dx-\sqrt{y}\,dy}=\sqrt{\dfrac{y^3}{x^3}}$ is given by

(a) $\frac{3}{2}\log\left(\frac{y}{x}\right)+\log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right|+\tan^{-1}\left(\frac{y}{x}\right)^{3/2}+c=0$
(b) $\frac{2}{3}\log\left(\frac{y}{x}\right)+\log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right|+\tan^{-1}\left(\frac{y}{x}\right)+c=0$
(c) $\frac{2}{3}\log\left(\frac{y}{x}\right)+\log\left(\frac{x+y}{x}\right)+\tan^{-1}\left(\frac{y^{3/2}}{x^{3/2}}\right)+c=0$
(d) None of the above

Section B : Numerical Type Questions

21 Sixteen metre of wire is available to fence off a flower bed in the form of a sector. If the flower bed has the maximum surface then radius is ....

22 A circle is drawn to pass through the extremities of the latusrectum of the parabola $y^2=8x$. It is given that, this circle also touches the directrix of the parabola. The radius of this circle is equal to ....

23 The number of permutations of the letters a, b, c, d such that b does not follow a, c does not follow b and d does not follow c, is ....

24 If $f'(x)=\sin(\log x)$ and $y=f\left(\dfrac{2x+3}{3-2x}\right)$ and $\dfrac{dy}{dx}$ at $x=1$ is equal to $p\sin\log(q)$, then value of $p+q$ is equal to ....

25 If $x + y + z = 1$ and x, y, z are positive numbers such that $(1-x)(1-y)(1-z) \geq kxyz$, then k^2 equals ………… .

26 If $f(x)$ is a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, then the value of n^4 is ………… .

27 Number of solutions of the equation $\sin x = [x]$, where [.] denotes the largest integer function, is

28 If the value of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is $\frac{m}{n} - \pi$, then value of mn is equal to

29 If the area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is $p\sqrt{3}$, then value of p is equal to

30 If $\hat{i} - 3\hat{j} + 5\hat{k}$ bisects the angle between $\hat{a}$ and $-\hat{i} + 2\hat{j} + 2\hat{k}$ when $\hat{a}$ is a unit vector, then $\hat{a} = \frac{\lambda\hat{i} + \mu\hat{j} + \gamma\hat{k}}{105}$, then the value of $\lambda^2 - 2\mu + v$ must be

Hints and Explanations

1 *(c)* Given equations can be rewritten as

$$x^2 + (y-1)^2 = 2^2$$

and $$(x-1)^2 + (y-2)^2 = 4^2$$

Here, centres are $C_1\ (0, 1)$ and $C_2\ (1, 2)$ and radii are

$$r_1 = 2, r_2 = 4.$$

Now, $$C_1C_2 = \sqrt{(1-0)^2 + (2-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\because$ $$|r_1 - r_2| = |2-4| = 2$$

$$C_1C_2 < |r_1 - r_2|$$

2 *(a)* Let $I = \int \sqrt{\frac{\cos 2x}{\sin^2 x}}dx$

$$= \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}dx$$

$$= \int \sqrt{\cot^2 x - 1}\, dx$$

On putting $\cot x = \sec\theta$

$\Rightarrow$ $-\text{cosec}^2 x\, dx = \sec\theta \tan\theta\, d\theta$,

we get

$$I = \int \sqrt{\sec^2\theta - 1} \times \frac{\sec\theta \tan\theta}{-\text{cosec}^2 x}d\theta$$

$$= -\int \frac{\sec\theta \tan^2\theta}{1 + \sec^2\theta}d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta}d\theta$$

$$= -\int \frac{1 - \cos^2\theta}{\cos\theta + \cos^3\theta}d\theta$$

$$= -\int \frac{(1 + \cos^2\theta) - 2\cos^2\theta}{\cos\theta(1 + \cos^2\theta)}d\theta$$

$$= -\int \sec\theta\, d\theta + 2\int \frac{d(\sin\theta)}{1 + \cos^2\theta}$$

$$= -\log|\sec\theta + \sqrt{\sec^2\theta - 1}|$$

$$+ \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2} + \sqrt{1 - \cos^2\theta}}{\sqrt{2} - \sqrt{1 - \cos^2\theta}}\right| + C$$

$$= -\log|\cot x + \sqrt{\cot^2 x - 1}|$$

$$+ \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}}\right| + C$$

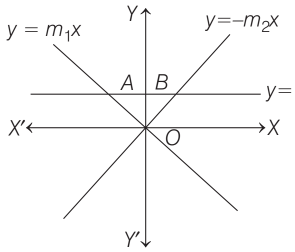
But $I = -\log|\cot x + \sqrt{\cot^2 x - 1}|$

$+ A + C$ [given]

$$\therefore\ A = \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}}\right|$$

3 *(a)* Since, m_1 and m_2 are the roots of the equation

$$x^2 + (\sqrt{3} + 2)x + (\sqrt{3} - 1) = 0.$$



$\therefore$ $$m_1 + m_2 = -(2 + \sqrt{3})$$

and $$m_1 m_2 = \sqrt{3} - 1$$

$\Rightarrow$ $$m_1 < 0,\ m_2 < 0$$

So, the point of intersections are

$A\left(\frac{1}{m_1}, 1\right)$ and $B\left(-\frac{1}{m_2}, 1\right)$

and $O(0, 0)$.

$$\therefore \text{Area of } \Delta OAB = -\frac{1}{2}\left\|\begin{vmatrix} \frac{1}{m_1} & 1 & 1 \\ \frac{-1}{m_2} & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}\right\|$$

$$= \frac{1}{2}\left|\frac{1}{m_1} + \frac{1}{m_2}\right| = \frac{1}{2}\left|\frac{m_1 + m_2}{m_1 m_2}\right|$$

$$= \frac{1}{2}\left(\frac{2 + \sqrt{3}}{\sqrt{3} - 1}\right)$$

4 *(c)* The general equation of the tangent to the ellipse is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \quad \text{...(i)}$$

Since, $y = \frac{3}{2}x - 10$ is a tangent to the ellipse, therefore its comparing with Eq. (i), we get

$$m = \frac{3}{2} \text{ and } a^2m^2 + b^2 = 100$$

$$\Rightarrow \quad 9a^2 + 4b^2 = 400 \quad \text{...(ii)}$$

Similarly, line

$$y = -\frac{1}{6}x + \frac{10}{3}$$

is a tangent to the ellipse, therefore its comparing with Eq. (i), we get

$$m = -\frac{1}{6}$$

and $$a^2m^2 + b^2 = \frac{100}{9}$$

$$\Rightarrow \quad a^2 + 36b^2 = 400 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii), we get

$$a^2 = 40$$

and $$b^2 = 10$$

Hence, required equation of the ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1.$$

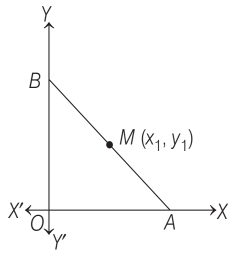
5 *(b)* Any line through the point of intersection of given lines is

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right) + \lambda\left(\frac{x}{b} + \frac{y}{a} - 1\right) = 0$$

$$x\left(\frac{1}{a} + \frac{\lambda}{b}\right) + y\left(\frac{1}{b} + \frac{\lambda}{a}\right) = (1 + \lambda)$$

$$\Rightarrow x\left(\frac{b + a\lambda}{ab}\right) + y\left(\frac{a + b\lambda}{ab}\right) = (1 + \lambda)$$

$$\Rightarrow \frac{x}{\left\{\frac{ab(1+\lambda)}{b + a\lambda}\right\}} + \frac{y}{\left\{\frac{ab(1+\lambda)}{a + b\lambda}\right\}} = 1$$



This meets the **X**-axis at

$$A \equiv \left(\frac{ab(1+\lambda)}{b+a\lambda}, 0\right)$$

and meets the **Y**-axis at

$$B \equiv \left(0, \frac{ab(1+\lambda)}{a+b\lambda}\right)$$

Let the mid-point of **AB** is $M(x_1, y_1)$. Then,

$$x_1 = \frac{ab(1+\lambda)}{2(b+a\lambda)}$$

and $$y_1 = \frac{ab(1+\lambda)}{2(a+b\lambda)}$$

$$\therefore \frac{1}{x_1} + \frac{1}{y_1} = \frac{2(b+a\lambda)}{ab(1+\lambda)} + \frac{2(a+b\lambda)}{ab(1+\lambda)}$$

$$= \frac{2}{ab(1+\lambda)}(b + a\lambda + a + b\lambda)$$

$$= \frac{2}{ab(1+\lambda)}(b+a)(1+\lambda)$$

$$\Rightarrow \frac{(x_1+y_1)}{x_1 y_1} = \frac{2(a+b)}{ab}$$

$$\Rightarrow 2x_1 y_1 (a+b) = ab(x_1 + y_1)$$

Hence, the locus of mid-point of **AB** is $2xy(a+b) = ab(x+y)$

6 *(b)* Since, $f(x)$ is continuous

$$\therefore \lim_{x\to 0}\left(\frac{2x - \sin^{-1}x}{2x + \tan^{-1}x}\right) = f(0)$$

$$\therefore f(0) = \lim_{x\to 0} \frac{\left(2 - \frac{1}{\sqrt{1-x^2}}\right)}{2 + \frac{1}{1+x^2}}$$

$$= \frac{2 - \frac{1}{\sqrt{1}}}{2 + \frac{1}{1}} = \frac{1}{3}$$

[∵ apply L'Hospital's Rule]

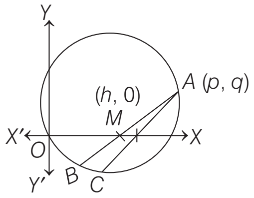
7 *(d)* Let **AB** be a chord of the circle through **A** (p,q) and **M** $(h, 0)$ be the mid-point of **AB.** Therefore, the coordinates of **B** are $(-p + 2h, -q)$.

Since, **B** lies on the circle $x^2 + y^2 = px + qy$, then

$$(-p+2h)^2 + (-q)^2 = p(-p+2h) + q(-q)$$

$$\Rightarrow 2p^2 + 2q^2 - 6ph + 4h^2 = 0$$

$$\Rightarrow 2h^2 - 3ph + (p^2 + q^2) = 0 \quad \ldots(i)$$



As, there are two distinct chords **AB** and **AC** from $A(p, q)$ which are bisected on **X**-axis there must be two distinct values of **h** satisfying Eq. (i), then **D** $= (b^2 - 4ac) > 0$, we have

$$(-3p)^2 - 4(2)(p^2 + q^2) > 0$$

$$\Rightarrow p^2 > 8q^2$$

8 *(c)* Since, $A^3 = B^3$ and $A^2B = B^2A$

$$\therefore A^3 - A^2B = B^3 - B^2A$$

$$\Rightarrow (A^2 + B^2)(A - B) = 0$$

$$\Rightarrow \det(A^2 + B^2)\det(A - B) = 0$$

9 *(b)* Here, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$$\therefore P(X=1) + P(X=3) + \ldots + P(X=49)$$

$$= {}^{100}C_1\left(\frac{1}{2}\right)^{100} + {}^{100}C_3\left(\frac{1}{2}\right)^{100} + \ldots + {}^{100}C_{49}\left(\frac{1}{2}\right)^{100}$$

$$= \left(\frac{1}{2}\right)^{100}({}^{100}C_1 + {}^{100}C_3 + \ldots + {}^{100}C_{49})$$

$$\left[\begin{array}{l} \because ({}^{100}C_1 + {}^{100}C_3 + \ldots + {}^{100}C_{99}) = 2^{99} \\ \text{but } {}^{100}C_{99} = {}^{100}C_1 \\ {}^{100}C_{97} = {}^{100}C_3, \ldots, {}^{100}C_{51} = {}^{100}C_{49} \\ \therefore 2({}^{100}C_1 + {}^{100}C_3 + \ldots + {}^{100}C_{49}) = 2^{99} \end{array}\right]$$

$$= \left(\frac{1}{2}\right)^{100} \times 2^{98} = \frac{1}{4}$$

10 *(c)* Line is parallel to plane as $(\hat{i} - \hat{j} + 4\hat{k})\cdot(\hat{i} + 5\hat{j} + \hat{k}) = 0$

General point on the line is $(\lambda + 2, -\lambda - 2, 4\lambda + 3)$. For $\lambda = 0$ point on this line is (2, –2, 3) and distance from $r\cdot(\hat{i} + 5\hat{j} + \hat{k}) = 5$ or $x + 5y + z = 5$, is

$$d = \frac{|2 + 5(-2) + 3 - 5|}{\sqrt{1 + 25 + 1}}$$

$$\Rightarrow d = \frac{|-10|}{3\sqrt{3}} = \frac{10}{3\sqrt{3}}$$

11 *(a)* Negation of $p \vee (\sim p \vee q)$

$$\Rightarrow \sim[p \vee (\sim p \vee q)] \equiv \sim p \wedge \sim(\sim p \vee q)$$

$$\equiv \sim p \wedge (\sim(\sim p) \wedge \sim q)$$

$$\equiv \sim p \wedge (p \wedge \sim q)$$

$$\equiv (p \wedge \sim q) \wedge \sim p$$

12 *(b)* Given,

$$f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5}, & x \neq 1 \\ -\dfrac{1}{3}, & x = 1 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{1}{2x-5}, & x \neq 1 \\ -\dfrac{1}{3}, & x = 1 \end{cases}$$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2(1+h) - 5} - \left(-\frac{1}{3}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2h-3} + \frac{1}{3}}{h}$$

$$= \lim_{h\to 0}\frac{3 + 2h - 3}{3h(2h-3)} = -\frac{2}{9}$$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2(1-h) - 5} - \left(-\frac{1}{3}\right)}{-h}$$

$$= \lim_{h\to 0} -\frac{2}{3(2h+3)} = -\frac{2}{9}$$

$$\therefore f'(1) = -\frac{2}{9}$$

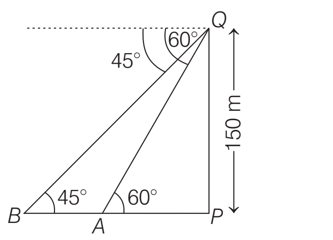
13 *(a)* $$\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\}$$

$$= \tan\left\{\pi - \cos^{-1}\left(\frac{2}{7}\right) - \frac{\pi}{2}\right\}$$

$$= \tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right)\right\} = \frac{2}{3\sqrt{5}}$$

14 *(b)* Let $PQ = 150$ m



In ΔAPQ, $\tan 60° = \frac{PQ}{AP}$

$$\Rightarrow AP = \frac{150}{\sqrt{3}} \quad \ldots(i)$$

and in ΔBPQ,

$$\tan 45° = \frac{PQ}{AB + AP}$$

$$\Rightarrow AB + \frac{150}{\sqrt{3}} = 150$$

$\Rightarrow \quad AB = \dfrac{150}{\sqrt{3}}(\sqrt{3}-1)$

$\therefore$ Speed of boat $= \dfrac{AB}{2}$

$= \dfrac{1}{2} \times \dfrac{150}{\sqrt{3}}(\sqrt{3}-1) \times 60$

$= \dfrac{4500}{\sqrt{3}}(\sqrt{3}-1)$ m/h

15 *(b)* Given equation of planes are

$4x + 4y - 5z = 12$ ...(i)

and $8x + 12y - 13z = 32$ ...(ii)

Let DR's of required line be (l, m, n).

From Eqs. (i) and (ii), we get

$4l + 4m - 5n = 0$

and $8l + 12m - 13n = 0$

$\Rightarrow \dfrac{l}{8} = \dfrac{m}{12} = \dfrac{n}{16}$

$\Rightarrow \dfrac{l}{2} = \dfrac{m}{3} = \dfrac{n}{4}$

Now, we take intersection point with $z = 0$ is given by

$4x + 4y = 12$ ...(iii)

and $8x + 12y = 32$ ...(iv)

On solving Eqs. (i) and (ii), we get the point (1, 2, 0).

$\therefore$ Required line is $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z}{4}$.

16 *(b)* Let $\Delta' = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$

$= abc + 2 - a - b - c > 0$

or $abc + 2 > a + b + c$ ...(i)

$\because$ AM > GM $\Rightarrow \dfrac{a+b+c}{3} > (abc)^{1/3}$

$a + b + c > 3(abc)^{1/3}$ ...(ii)

From Eqs. (i) and (ii),

$abc + 2 > 3(abc)^{1/3}$

Let $(abc)^{1/3} = x$

Then, $x^3 + 2 > 3x$

$\Rightarrow (x-1)^2(x+2) > 0$

$\therefore x + 2 > 0 \Rightarrow x > -2$

$\Rightarrow x^3 > -8 \Rightarrow abc > -8$

17 *(c)* Here, $P(W) = \dfrac{a}{a+b}$

and $P(B) = \dfrac{b}{a+b}$

$\therefore$ Probability of **A** winning

$= P(W) + P(\overline{W})P(\overline{B})P(W) + \dots$

$= \dfrac{P(W)}{1 - P(\overline{W})P(\overline{B})}$

$= \dfrac{\dfrac{a}{a+b}}{1 - \dfrac{b}{a+b} \cdot \dfrac{a}{a+b}}$

$= \dfrac{a(a+b)}{a^2 + b^2 + ab} = P_1$ [say]

and probability of **B** winning

$= 1 - P_1 = 1 - \dfrac{a^2 + ab}{a^2 + b^2 + ab}$

$= \dfrac{b^2}{a^2 + b^2 + ab} = P_2$ [say]

Given, $P_1 = 2P_2$

$\Rightarrow \dfrac{a^2 + ab}{a^2 + b^2 + ab} = \dfrac{2b^2}{a^2 + b^2 + ab}$

$\Rightarrow a^2 + ab - 2b^2 = 0$

$\Rightarrow (a - b)(a + 2b) = 0$

$\Rightarrow a - b = 0 \quad [\because a + 2b \neq 0]$

$\Rightarrow a = b$

$\therefore a : b = 1 : 1$

18 *(d)* $I + f + f'$

$= (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n$

$= 2k$ [even integer]

$\therefore f + f' = 1$

Now,

$(I + f)f' = (5 + 2\sqrt{6})^n (5 - 2\sqrt{6})^n$

$= (1)^n = 1$

$\Rightarrow (I + f)(1 - f) = 1$

$\Rightarrow I = \dfrac{1}{1-f} - f$

19 *(d)* Since, $n = 11$, then median term

$= \left(\dfrac{11+1}{2}\right)$th term = 6 th term

As, last five observations are increased by 4. Hence, the median of the 6th observations will remain same.

20 *(d)* We have, $\dfrac{\sqrt{x}\,dx + \sqrt{y}\,dy}{\sqrt{x}\,dx - \sqrt{y}\,dy} = \sqrt{\dfrac{y^3}{x^3}}$

$\Rightarrow \dfrac{d(x^{3/2}) + d(y^{3/2})}{d(x^{3/2}) - d(y^{3/2})} = \dfrac{y^{3/2}}{x^{3/2}}$

$\Rightarrow \dfrac{du + dv}{du - dv} = \dfrac{v}{u}$

where $u = x^{3/2}$ and $v = y^{3/2}$

$\Rightarrow u\,du + u\,dv = v\,du - v\,dv$

$\Rightarrow u\,du + v\,dv = v\,du - u\,dv$

$\Rightarrow \dfrac{u\,du + v\,dv}{u^2 + v^2} = \dfrac{v\,du - u\,dv}{u^2 + v^2}$

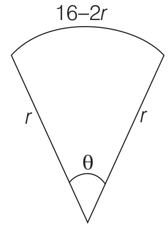
$\Rightarrow \dfrac{d(u^2 + v^2)}{u^2 + v^2} = -2d\,\tan^{-1}\left(\dfrac{v}{u}\right) + c$

On integrating, we get

$\log(u^2 + v^2) = -2\tan^{-1}\left(\dfrac{v}{u}\right) + c$

$\Rightarrow \dfrac{1}{2}\log(x^3 + y^3) + \tan^{-1}\left(\dfrac{y}{x}\right)^{3/2} = \dfrac{c}{2}$

21 *(4)* Let **r** be the radius of the sector and angle subtended at the centre be θ.



Then, **S** = surface area of sector

$= \dfrac{\theta}{360} \times \pi r^2$

We know, $\theta = \dfrac{\text{length of arc}}{\text{radius}}$

$= \dfrac{16 - 2r}{r}$

$\therefore \quad S = \dfrac{\theta}{2\pi}\pi r^2 = \dfrac{16-2r}{2r} \cdot r^2$

$\Rightarrow \quad S = (8 - r) \cdot r = 8r - r^2$

$\Rightarrow \quad \dfrac{dS}{dr} = 8 - 2r$

Now, for area to be maximum,

$\dfrac{dS}{dr} = 0$

$\Rightarrow \quad r = 4$

22 *(4)* Extremities of the latusrectum of the parabola are (2, 4) and (2, – 4).

Since, any circle drawn with any focal chord at its diameter touches the directrix, thus equation of required circle is

$(x - 2)^2 + (y - 4)(y + 4) = 0$

$\Rightarrow x^2 + y^2 - 4x - 12 = 0$

$\therefore$ Radius $= \sqrt{(2)^2 + 12} = 4$

23 *(11)* We have the following cases.

Case I	*a*			

In this case we have only two possibilities, namely, **a c bd** and **a d c b**

Case II	*b*			

In this case we have only three possibilities, namely, **ba d c**, **bd a c** and **bd c a**

Case III	*c*			

In this case we have only three possibilities, namely, **c a d b**, **c bd a** and **c ba d**

Case IV	*d*			

In this case we have only three possibilities, namely, **d c b a**, **d a c b** and **d b a c**

Hence, the total number of ways

$$= 2 + 3 + 3 + 3 = 11$$

24 *(17)* Given that, $y = f\left(\frac{2x+3}{3-2x}\right)$

$$\Rightarrow \frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right)\frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$$

$$= \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$$

$$\left[\frac{(3-2x)(2)-(2x+3)(-2)}{(3-2x)^2}\right]$$

$$= \frac{12}{(3-2x)^2}\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x=1)} = \frac{12}{(3-2)^2}\sin\log(5)$$

$$= 12\sin\log(5)$$

Here, $p = 12, q = 5$

$\therefore \quad p + q = 17$

25 *(64)* Clearly, $\frac{x+y}{2} \geq \sqrt{xy}; \frac{y+z}{2} \geq \sqrt{yz}$

and $\frac{x+z}{2} \geq \sqrt{xz}$

$$\therefore \frac{(x+y)}{2}\cdot\frac{(y+z)}{2}\cdot\frac{(x+z)}{2} \geq \sqrt{xy}\cdot\sqrt{yz}\cdot\sqrt{xz}$$

$$\Rightarrow (1-z)(1-x)(1-y) \geq 8xyz \quad [\because x + y + z = 1]$$

$\therefore \quad k = 8$

Hence, $\quad k^2 = 64$

26 *(256)* $\quad f(x) = f(1 + 1 + 1 + \ldots + x$ times)

$$= f(1)f(1)f(1)\ldots\ldots x \text{ times}$$

$$= [f(1)]^x = 3^x$$

$$\therefore \sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} 3^x = 3^1 + 3^2 + \ldots + 3^n$$

$$= \frac{3^1 - 3^n \cdot 3}{1-3}$$

$$= \frac{3^{n+1}-3}{2} \quad \left[\because \text{sum} = \frac{a - lr}{1-r}\right]$$

$$\therefore \frac{3^{n+1}-3}{2} = 120 \Rightarrow 3^{n+1} = 243 = 3^5$$

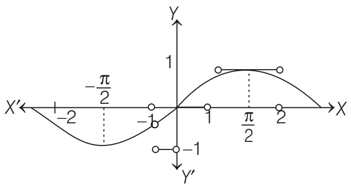
$\Rightarrow \quad n + 1 = 5$

$\Rightarrow \quad n = 4$

$\therefore \quad n^4 = 256$

27 *(2)* $y = \sin x = [x]$

Graphs of $y = \sin x$ and $y = [x]$ are as shown.



Hence, two solutions are $x = 0$ and $x = \frac{\pi}{2}$.

28 *(154)* Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$

$$= \int_0^1 x^4\left(x^2 - 4x + 5 - \frac{4}{1+x^2}\right)dx$$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4)\,dx - 4\int_0^1 \frac{x^4}{(1+x^2)}dx$$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4)\,dx - 4\int_0^1 \left(x^2 - 1 + \frac{1}{1+x^2}\right)dx$$

$$= \left(\frac{x^7}{7} - \frac{4x^6}{6} + x^5\right)_0^1 - 4\left(\frac{x^3}{3} - x + \tan^{-1}x\right)_0^1$$

$$= \left(\frac{1}{7} - \frac{4}{6} + 1\right) - 4\left(\frac{1}{3} - 1 + \frac{\pi}{4}\right)$$

$$= \frac{22}{7} - \pi$$

$m = 22, n = 7$

Hence, $m \times n = 22 \times 7 = 154$

29 *(12)* Given parabolas are

$$5x^2 - y = 0$$

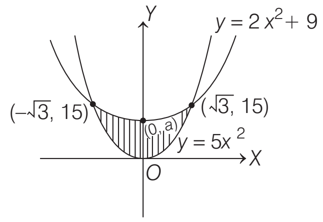
and $\quad 2x^2 - y + 9 = 0$

Now, eliminating y from above equations, we get

$$5x^2 - (2x^2 + 9) = 0$$

$$\Rightarrow 3x^2 = 9 \Rightarrow x = \pm\sqrt{3}$$

Given parabolas intersect at $(\sqrt{3}, 15)$ and $(-\sqrt{3}, 15)$.



The two parabolas are

$$x^2 = \frac{1}{5}y, \; x^2 = \frac{1}{2}(y - 9)$$

$$\therefore \text{Area} = 2\int_0^{\sqrt{3}} (y_1 - y_2)\,dx$$

$$= 2\int_0^{\sqrt{3}} [(2x^2 + 9) - 5x^2]\,dx$$

$$= 2\int_0^{\sqrt{3}} (9 - 3x^2)\,dx$$

$$= 2[9x - x^3]_0^{\sqrt{3}}$$

$\therefore$ Area $= 12\sqrt{3}$

$\therefore \quad p = 12$

30 *(1817)* We must have

$$\alpha(\hat{i} - 3\hat{j} + 5\hat{k}) = \hat{a} + \frac{2\hat{k} + 2\hat{j} - \hat{i}}{3}$$

$$\Rightarrow 3\hat{a} = \alpha(3\hat{i} - 9\hat{j} + 15\hat{k}) - 2\hat{k} - 2\hat{j} + \hat{i}$$

$$\Rightarrow 3\hat{a} = (3\alpha + 1)\hat{i} - (9\alpha + 2)\hat{j} + (15\alpha - 2)\hat{k}$$

$$\therefore 3|\hat{a}| = |(3\alpha + 1)\hat{i} - (9\alpha + 2)\hat{j} + (15\alpha - 2)\hat{k}|$$

$$\Rightarrow 3 = \sqrt{\{(3\alpha + 1)^2 + (9\alpha + 2)^2 + (15\alpha - 2)^2\}}$$

$$\Rightarrow 9 = 135\alpha^2 - 18\alpha + 9$$

$$\therefore \alpha = 0, \frac{2}{35}$$

For $\alpha = 0, \hat{a} = \frac{\hat{i} - 2\hat{j} - 2\hat{k}}{3}$

(not acceptable)

For $\alpha = \frac{2}{35}, \hat{a} = \frac{2}{35}(\hat{i} - 3\hat{j} + 5\hat{k}) - \frac{(2\hat{k} + 2\hat{j} - \hat{i})}{3}$

$$= \frac{1}{105}(41\hat{i} - 88\hat{j} - 40\hat{k})$$

On comparing, we get

$\lambda = 41, \mu = -88, v = -40$

$\therefore \lambda^2 - 2\mu + v = (41)^2 + 2(88) - 40$

$= 1681 + 176 - 40 = 1817$